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The inverse scattering problem in linear elasticity for few incident waves using nonlinear integral equations

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Abstract

We consider the inverse scattering problem of determining the shape of a rigid scatterer, a cavity or an inclusion in a homogeneous and isotropic two dimensional elastic medium using few incident time-harmonic fields. We present some general uniqueness and local uniqueness results for the inverse scattering problem for a rigid obstacle, based on the Faber Krahn inequality. The inverse problem is solved using nonlinear and ill-posed integral equations for the unknown boundary derived from the integral representation of the scattered field. A linearization of the equations is performed and Tikhonov regularization is used to solve approximately the equations. Numerical examples are given which illustrate the applicability of the method and show some characteristics of the inverse elastic scattering process.

Key words: Inverse scattering problem, linear elasticity.
Field: Inverse problems.
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1 Direct Scattering Problem

The direct elastic scattering problem is to find the scattered field u^{sc} from the knowledge of the obstacle D with smooth boundary Γ and the incident wave u^{inc}. In this work we assume that the environment is a two-dimensional linear homogeneous and isotropic elastic medium with Lamé constants μ and λ, satisfying $\lambda+\mu > 0$ and $\mu > 0$ and density ρ and that the scatterer is either a rigid obstacle, or a cavity or an inclusion with different Lamé constants from the exterior domain. The displacement field u^{sc} of the scattered wave satisfies the Navier equation

$$\left(\Delta^* + \rho\omega^2\right) u^{sc} = 0 \;\; in \;\; R^2\backslash\bar{D} \tag{1}$$

where $\Delta^* u := \mu\Delta u + (\lambda+\mu)\nabla(\nabla\cdot u)$ and $\omega > 0$ is the circular frequency. Any solution to (1) can be decomposed as $u := u_p + u_s$ where u_p is a longitudinal and u_s a transversal wave, defined by $u_p := -\frac{1}{k_p^2}\nabla(\nabla\cdot u)$ and $u_s := u - u_p$. The wavenumbers are given by $k_p^2 := \rho\omega^2/(\lambda+2\mu)$ and $k_s^2 := \rho\omega^2/\mu$ respectively. The total displacement field u^{tot} is given by $u^{tot} = u^{sc} + u^{inc}$ where u^{inc} is an entire solution to the Navier equation. The incident field is either a longitudinal or a transversal wave. For a rigid scatterer, the total displacement field u^{tot} has to satisfy the Dirichlet boundary condition

$$u^{sc} + u^{inc} = 0 \;\; on \;\; \Gamma. \tag{2}$$

For a cavity, u^{tot} has to satisfy the Neumann boundary condition

$$T\left(u^{sc} + u^{inc}\right) = 0 \;\; on \;\; \Gamma \tag{3}$$

Here Tu is the traction operator on Γ and is defined by $Tu := \lambda(\nabla\cdot u)\nu + 2\mu\,(\nu\cdot\nabla u) + \mu(\nabla\cdot(Qu))Q\nu$ where ν is the unit normal vector and Q denotes the unitary matrix

$$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For the inclusion problem, we have u^{sc} and an interior field u^{int} satisfying

$$\left(\Delta_e^* + \rho_e\omega^2\right)u^{sc} = 0 \; in \; R^2\backslash\bar{D} \; and \; \left(\Delta_i^* + \rho_i\omega^2\right)u^{int} = 0 \; in \; D \tag{4}$$

and the boundary conditions

$$u^{int} = u^{sc} + u^{inc}\;,\; T_i u^{int} = T_e\left(u^{sc} + u^{inc}\right) on \;\; \Gamma \tag{5}$$

where Δ_e^*, T_e and Δ_i^*, T_i depend on the Lamé constants λ_e, μ_e in the exterior domain of D and on λ_i, μ_i in the interior of D, respectively. To ensure uniqueness in all three cases, u^{sc} is required to satisfy the Kupradze radiation condition:

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial u_p}{\partial r} - ik_p u_p\right) = 0, \quad \lim_{r\to\infty}\sqrt{r}\left(\frac{\partial u_s}{\partial r} - ik_s u_s\right) = 0, \quad r = |x| \tag{6}$$

uniformly over all directions. The fundamental solution for the Navier equation is given by

$$\Phi(x,y) \;\; := \;\; \frac{i}{4\mu}H_0^{(1)}(k_s r)I + \frac{i}{4\omega^2}\nabla_x\nabla_x^T\left[H_0^{(1)}(k_s r) - H_0^{(1)}(k_p r)\right] \tag{7}$$

where $r = |x-y|$ and $H_0^{(1)}$ is the Hankel function of order zero and of the first kind. Then the single-layer potential with vector density φ on Γ is defined by

$$(S\varphi)(x) := \int_\Gamma \Phi(x,y)\varphi(y)ds(y), \;\; x\in\Gamma \tag{8}$$

and the double-layer potential is defined by

$$(D\varphi)(x) := \int_\Gamma \left[T_y\Phi(x,y)\right]^T\varphi(y)ds(y), \;\; x\in\Gamma \tag{9}$$

For all three boundary value problems the representation of the solution as a certain combination of a single- and a double-layer potential transforms the direct problem to a uniquely solvable integral equation for the density φ, equivalently. For more details, we refer to [2] for the case of a rigid scatterer, to [6] for a cavity and to [8, 9] for the inclusion problem. From the asymptotic behavior of the Hankel function $H_0^{(1)}$ and $H_1^{(1)}$ we obtain that the far fields patterns of the combined single- and double-layer potential for the case of the rigid scatterer are given by

$$u_j^{\infty}(\hat{x}) = -\gamma_j i \int_{\Gamma} J^*(\hat{x}) \left[F(\hat{x}, y) + \frac{\eta}{k_j} I \right] e^{-ik_j \hat{x}\cdot y} \varphi(y) ds(y) \tag{10}$$

and for the cavity are given by

$$u_j^{\infty}(\hat{x}) = \gamma_j \int_{\Gamma} J^*(\hat{x}) \left[\eta F(\hat{x}, y) + \frac{1}{k_j} I \right] e^{-ik_j \hat{x}\cdot y} \varphi(y) ds(y) \tag{11}$$

for $j = p, s$ where

$$\gamma_j = \begin{cases} \frac{e^{\frac{i\pi}{4}}}{\lambda+2\mu} \sqrt{\frac{k_p}{8\pi}} & \text{if } j = p \\ \frac{e^{\frac{i\pi}{4}}}{\mu} \sqrt{\frac{k_s}{8\pi}} & \text{if } j = s \end{cases}, \quad J^*(\hat{x}) = \begin{cases} J(\hat{x}) & \text{if } j = p \\ I - J(\hat{x}) & \text{if } j = s \end{cases}$$

and $F(\widehat{x}, y) = \lambda \widehat{x} \nu(y)^T + \mu \nu(y) \widehat{x}^T + \mu \nu(y) \cdot \widehat{x} I$. Similar forms are derived for the inclusion problem. The knowledge of the two corresponding far fields is enough to describe the scattering process.

2 Inverse Scattering Problem

The inverse elastic scattering problem is to recover the geometry of the scatterer D from the knowledge of the pair of p and s far field patterns: $u(\widehat{x}) := (u_p^{\infty}(\widehat{x}), u_s^{\infty}(\widehat{x}))$ for all $\widehat{x} \in \Omega$, where Ω is the unit ball, for one or few incident directions d and one frequency ω. To face this geometrical inverse problem we use the idea of the pair of nonlinear integral equations introduced in [7] for the Laplace equation. In general, uniqueness for the inverse scattering problem for one incident wave is an open problem for all three cases in elasticity. However, we can prove a local uniqueness result for the Dirichlet problem only. For the theorem we will need the class of domains S, such that if $D_1, D_2 \in \mathcal{S}$, then

$$0 < E\left(D^*\right) < \frac{\mu\, \pi\, k_{01}^2}{\rho\, \omega^2} \tag{12}$$

and $\overline{\mathbb{R}^2 \backslash (D_1 \cup D_2)}$ is connected. In the above inequality, $D^* = D_1 \cup D_2 - \overline{D_1 \cap D_2}$, $E\left(D^*\right)$ is the area of D^* and k_{01} is the first zero of the Bessel function $J_0\left(k\right)$.

Theorem 1 *Assume that D_1 and D_2 are two rigid scatterers and D_1, $D_2 \in \mathcal{S}$ such that the far field patterns $u_1^{\infty}\left(\widehat{x},d\right)$ and $u_2^{\infty}\left(\widehat{x},d\right)$ coincide for all $\widehat{x} \in \Omega$, for one incident plane wave with direction of incidence d and for fixed wave numbers. Then $D_1 = D_2$.*

Proof
The main argument about the difference set and the Faber Krahn inequality in acoustics is proposed in [3]. It is well known that for any bounded domain $B \subset \mathbb{R}^2$ if $w \in H^1(B) \cap C^1(B) \cap C(\bar{B})$ and satisfies $w = 0$ on ∂B then $w \in H_0^1(B)$. Following Schiffer's argument, assume that D_1 and D_2 are two domains producing the same far fields, then the total field u^{tot} is an eigenfunction of the Lamé operator in their difference set D^*. If we use in the last argument as B the set D^* then $u^{tot} \in H_0^1(D^*)$. Dividing by μ the Lamé operator takes the form $\Delta^* = \Delta + a\, \nabla\nabla\cdot$, where $a = \frac{(\lambda+\mu)}{\mu}$. If $\Lambda_n(D^*)$ are the corresponding eigenvalues then $\frac{\rho\,\omega^2}{\mu}$ is also an eigenvalue of $-\Delta^*$, that is $\Lambda_1(D^*) \leq \frac{\rho\,\omega^2}{\mu}$. Using the Faber - Krahn inequality we can find the following lower bound estimate [5]: $\frac{\pi\, k_{01}^2}{E(D^*)} \leq \lambda_1 \leq \Lambda_1(D^*)$. So, we deduce $\Lambda_1(D^*) \leq \frac{\rho\,\omega^2}{\mu}$, and $\frac{\mu\ \pi\, k_{01}^2}{\rho\ \omega^2} \leq E(D^*)$. But if $D_1, D_2 \in \mathcal{S}$ we infer that $E(D^*) < \frac{\mu\ \pi\, k_{01}^2}{\rho\ \omega^2}$ which is a contradiction. □

From here on we assume that our scatterers belong to class $\mathcal{S}$. This provides uniqueness for the Dirichlet problem only. For $h = Tu|_\Gamma$ we introduce the operator

$$(Sh)(x) = \int_\Gamma \Phi(x,y)h(y)ds(y), \;\; x \in R^2\backslash\Gamma \tag{13}$$

Then the unknown boundary curve Γ and the unknown traction h of the corresponding solution u satisfy the 2 by 2 system of equations

$$Sh = -u^{inc}\big|_\Gamma \tag{14}$$

and

$$S^\infty h = u^\infty \tag{15}$$

where $S^\infty h$ is the pair of far fields from the single layer representation of u^{sc} and u_∞ is the pair of measured p and s far fields. Conversely, if Γ and h satisfy (14) and (15) then Γ solves the inverse problem. For the case of a cavity, for $h = u|_\Gamma$ we introduce the operators

$$(Dh)(x) = \int_\Gamma \left[T_y\Phi(x,y)\right]^T h(y)ds(y), \;\; x \in R^2\backslash\Gamma \tag{16}$$

$$(L\varphi)(x) := T_x \int_\Gamma \left[T_y\Phi(x,y)\right]^T h(y)ds(y), \;\; x \in R^2\backslash\Gamma \tag{17}$$

Then the unknown boundary curve Γ and the unknown solution h satisfy the 2 by 2 system of equations

$$Lh = -Tu^{inc}\big|_\Gamma \tag{18}$$

and

$$D^\infty h = u^\infty \tag{19}$$

where $D^\infty h$ is the pair of far fields from the double layer representation of u^{sc}. Conversely, if Γ and h satisfy (18) and (19) then Γ solves the inverse problem. Both inverse obstacle scattering problems and the 2 by 2 systems of integral equations are equivalent. Similar and more complicated forms and equivalent system is derived for the case of inclusions. Due to the non-linearity and the ill-posedness of the integral equation systems *Fréchet* derivatives of the operators and Tikhonov regularization are needed. We assume the following parametrization for the boundary $\Gamma = \{x(t) = r(t)(cost, sint)\,,\, t \in [0, 2\pi]\}$ and then we obtain the Fréchet derivatives of the operators by formally differentiating their kernels with respect to r. Both systems (14)-(15) and (18)-(19) can be presented as

$$A(r, \varphi) = w(r) \tag{20}$$

and

$$C^{\infty}(r, \varphi) = w_{\infty} \tag{21}$$

where A, C are integral operators depending on each scattering problem, w is the corresponding right hand side and $\varphi = |x'|(h \circ x)$. In general, there are two iterative schemes for solving this integral equation system. The first one is to linearize both equations and then solve for the density φ and the radial function r. The second one is to solve first (20) or (21) for φ and then keeping φ fixed to solve the other linearized integral equation to obtain the update for r. Here we make use of the second iterative scheme because the computational cost is smaller solving two small linear systems than solving a full linearized system.

3 Numerical Results

For the numerical implementation, we approximate the radial function by a trigonometric polynomial [4, 7] and we use the trapezoidal rule for the integrals with smooth kernels. For those with singular and hypersingular kernels we replace the trapezoidal rule by convergent quadrature rules based on trigonometric interpolation [1]. We denote by l_0 and l_1 the regularization parameters for penalizing q and h respectively. In both examples we take $d = (1, 0)$, $\rho = 1$ and $\eta = 1$. For the case of a rigid scatterer we consider a kite - shaped boundary and u^{inc} is a longitudinal wave. Reconstructions with $\mu = 1.5$ and $\lambda = 2.5$ Lamé constants, $n = 64$ collocation points, $\omega = 3.5$ circular frequency, $m = 9$ Fourier coefficients, $l_1 = 10^{-3}$ and $l_0 = 10^{-2}$ are shown in figure 1. For the case of a cavity we consider a peanut - shaped boundary and u^{inc} is a transversal wave. Reconstructions with $\mu = 1.5$, $\lambda = 4$, $n = 32$, $\omega = 5$, $m = 4$, $l_1 = 10$ and $l_0 = 10^3$ are shown in figure 2.

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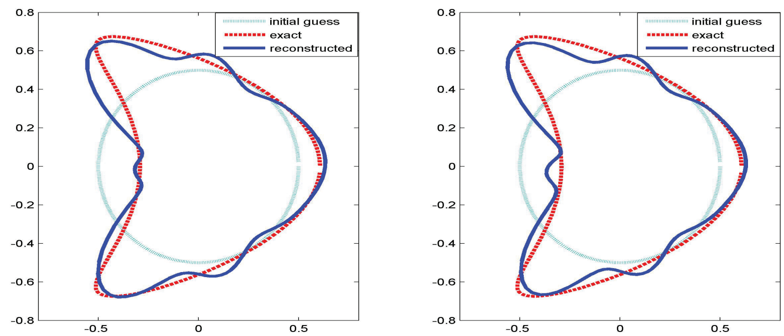

Figure 1: Reconstructions with 6 iterations, no noise (left) and 4% (right).

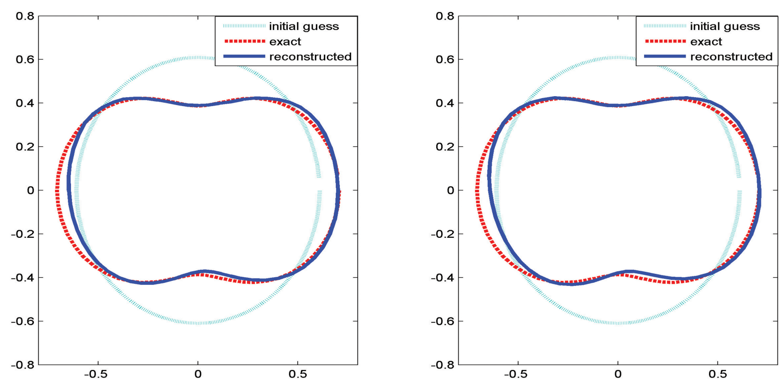

Figure 2: Reconstructions with 8 iterations, no noise (left) and 6% (right).

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