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THE DETECTION OF AN INCLUSION IN 2-D LINEAR ELASTICITY USING NON-LINEAR INTEGRAL EQUATIONS

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1 INTRODUCTION

In this work we consider the inverse scattering problem of determining an inclusion in two-dimensional linear elasticity. This problem is solved using the non-linear integral equations method, introduced in [4] for the Laplace equation. This method has been implemented by Kress and Ivanyshyn for the inverse scattering problem by an inclusion and a crack in acoustics. In elasticity the idea of the pair of non-linear integral equations adopted in [1] for detecting rigid scatteres and cavities.

Let D_i denote the inclusion, a bounded domain with smooth boundary Γ, and D_e the unbounded exterior domain. Both domains are filled with different isotropic and homogeneous elastic mediums with Lamé constants $\lambda_\alpha, \mu_\alpha$ and densities ρ_α, where $\alpha = i, e$. The scattering of the incident field u^{inc} produces two fields, the scattered field u^e and the interior field u^i. Each of them satisfies the Navier equation in D_α and certain boundary conditions. Due to the Helmholtz decomposition, each field u^α can be written as $u^\alpha = u_p^\alpha + u_s^\alpha$, where u_p^α is a longitudinal and u_s^α is a transversal wave. The incident field is either a longitudinal plane wave or a transversal plane wave.

Solving the direct scattering problem and using the integral representation of u^e and the asymptotic behavior of any solution to the Navier equation yields the explicit form of the pair of the far field patterns $u^\infty = (u_p^\infty, u_s^\infty)$ which is necessary to formulate the inverse scattering problem. That is to reconstruct the boundary Γ from the knowledge of u^∞ for all $\hat{x} = x/|x|$ and for one incident plane wave.

2 THE METHOD OF NON-LINEAR INTEGRAL EQUATIONS

In linear elasticity, the method proposed by Kress and Rundell, arises from Betti's integral theorems and the integral representation of the solutions. More precisely, applying Betti's third formula we obtain a representation for each displacemnt field u^α as a combination of an elastic single- and double-layer potential. Then we apply the boundary conditions and the well-known jump relations of the potentials on Γ to arrive at a 2×2 system of integral equations for three unknowns, two densities (h, g) and the boundary. The two densities correspond to $u|_\Gamma$ and $T_e u|_\Gamma$ respectively, where $u = u^e + u^{inc}$. The third equation arises from the far field patterns. Finally, we choose the following parametrization for the boundary curve $\Gamma = \{x(t) = r(t)(cost, sint)\}$, $t \in [0, 2\pi]$ with 2π periodic C^2 smooth functions

$x : R \to R^2$ and we can set $\psi = |x^{'}|(h \circ x)$ and $\phi = |x^{'}|(g \circ x)$. With those notations the 3×3 system of integral equations is transformed into

$$\mathcal{A}(r, \psi) + \mathcal{B}(r, \phi) = \mathcal{C} \tag{1}$$

where $\mathcal{A} = \big(\mathcal{A}_1\ \mathcal{A}_2\ \mathcal{A}_\infty\big)^T$, $\mathcal{B} = \big(\mathcal{B}_1\ \mathcal{B}_2\ \mathcal{B}_\infty\big)^T$ and $\mathcal{C} = \big(\mathcal{C}_1(r)\ \mathcal{C}_2(r)\ \mathcal{C}_\infty(\hat{x})\big)^T$. Here $\mathcal{A}_k, \mathcal{B}_k : [L^2([0, 2\pi])]^2 \to [L^2([0, 2\pi])]^2$ are combinations of parameterised elastic single- and double-layer operators and their tractions and have to be considered as acting on the second variable. The first variable indicates the dependence of the operators on x. The right hand side depends on the parameterised u^{inc} and u^∞.

The system (1) is linear with respect to the densities but non-linear with respect to the boundary Γ. This leads to linearization of the integral operators using Fréchet derivatives with respect to r. In order to overcome the ill-posedness of the linearized system due to the presence of compact operators we apply Tikhonov regularization. For solving system (1) we apply the following iterative scheme:

(i) Given an initial guess for r we find the densities from the first two equations.

(ii) We linearize the third equation with the far field patterns using Fréchet derivatives with respect to r in direction q and solve it keeping the densities fixed to obtain the update $r + q$ for the radial function.

(iii) The procedure continues until a suitable stopping criteria is satisfied.

3 CONVERGENCE RESULTS FOR THE RELATED NEWTON-TYPE METHOD

The iterative scheme adopted before can be seen as a Newton-type method. The first iteration of the algorithm is related to the first step of a Newton-type scheme. However the main advantage of the method presented in this work is that no solution of the direct problem is required in each iteration step. Following the results by Hettlich [2] for the Helmholtz equation we prove that the Fréchet derivative is an operator mapping Γ into far field patterns of an inclusion problem with different boundary conditions depending on the pertubation of the boundary. This characterization is needed to prove the source condition [3] for the Fréchet derivative which is necessary for an optimal convergence rate of the form

$$\|r_N - r\| = O((-ln\delta)^{-p}), \text{ as } \delta \to 0, \tag{2}$$

where N denotes the stopping index and $p > 0$. Finally, numerical results and boundary reconstructions are presented to show the applicability of the proposed method.

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